A ROKHLIN CONJECTURE AND SMOOTH QUOTIENTS BY THE COMPLEX CONJUGATION OF SINGULAR REAL ALGEBRAIC SURFACES

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§1 Introduction

1.1. A Rokhlin Conjecture and its variations. Considering real algebraic varieties, I use prefixes $\mathbb{C}$ and $\mathbb{R}$ to denote their complex point sets and the real point sets respectively, and put a bar to denote the orbit space for the involution of the complex conjugation, conj, for example, $\mathbb{C}X$, $\mathbb{R}X$, $\overline{X} = \mathbb{C}X/\operatorname{conj}$. I identify $\mathbb{R}X$ with its image in $\overline{X}$. The same conventions will be used in the notation of conj-invariant subsets of a real algebraic variety $\mathbb{C}X$.

If $\mathbb{C}X$ is a non-singular complex surface, then $\overline{X}$ is a 4-manifold and the quotient map $q\colon \mathbb{C}X \to \overline{X}$ is a double covering branched along $\mathbb{R}X$. One can endow $\overline{X}$ with an orientation and a smooth structure making q smooth and orientation preserving, for example $\overline{\mathrm{P}}^2 = \mathbb{C}\mathrm{P}^2/\operatorname{conj}$ is well known to be diffeomorphic to S^4.

In 1980's I learned from V.A.Rokhlin several problems about the topology of $\overline{X}$ for double planes $\mathbb{C}X$, motivated by studying the topology of their branching loci, the curves $\mathbb{C}A \subset \mathbb{C}\mathrm{P}^2$. One of these problems is as follows. A curve $\mathbb{C}A \subset \mathbb{C}\mathrm{P}^2$, defined by a real form $f(x, y, z)$ of degree $d = 2k$, with $\mathbb{R}A \neq \varnothing$, divides $\mathbb{R}\mathrm{P}^2$ into two regions, $\mathbb{R}\mathrm{P}^2_{\pm} = \{[x : y : z] \in \mathbb{R}\mathrm{P}^2 \mid \pm f \geqslant 0\}$. If $\mathbb{R}A$ is non-singular, then one of these regions is orientable, the other is not, and for the sake of definiteness it is usually assumed that $\mathbb{R}\mathrm{P}^2_{+}$ is the orientable region.

The unions $\mathfrak{A}_{\pm} = \mathbb{R}\mathrm{P}^2_{\pm} \cup \overline{A}$ are closed surfaces in $S^4 = \overline{\mathrm{P}}^2$, called by Rokhlin *the Arnold surfaces*. They consist of the two smooth pieces, $\mathbb{R}\mathrm{P}^2_{\pm}$ and $\overline{A}$, which meet normally along $\mathbb{R}A$, and therefore $\mathfrak{A}_{\pm}$ can be smoothed along $\mathbb{R}A$. Rokhlin suggested to prove that the embeddings $\mathfrak{A}_{\pm} \subset S^4$ are *standard*, in the sense that they can be obtained via ambient connected sum from a few copies of a standard embedding of a torus, $T^2 \subset S^4$, if $\mathfrak{A}_{\pm}$ is orientable, or from standard embeddings of $\mathbb{R}\mathrm{P}^2$ if $\mathfrak{A}_{\pm}$ is not orientable (recall that there exist two isotopy classes of standard embeddings of $\mathbb{R}\mathrm{P}^2$ into S^4, which are distinguished by the normal Euler number, 2 or -2).

The surfaces $\mathfrak{A}_{\pm}$ appeared in [A1] in connection with studying the topology of the double planes $p\colon X \to \mathbb{C}\mathrm{P}^2$ branched along $\mathbb{C}A$. There exist 2 liftings of the complex conjugation from $\mathbb{C}\mathrm{P}^2$ to X, and I denote them by $\operatorname{conj}^{\pm}$. The double plane X endowed with an involution $\operatorname{conj}^{\pm}$ is denoted by $\mathbb{C}X^{\pm}$. The reason behind such a notation is that one can view $\mathbb{C}X^{\pm}$ as a real algebraic surface defined by the equation $w^2 = \pm f(x, y, z)$ in a weighted projective 3-space. As follows from [Ar], the projection $q^{\pm}\colon \overline{X}^{\pm} \to S^4 = \overline{\mathrm{P}}^2$ induced by p is a double covering branched

along $\mathfrak{A}_{\mp}$. Thus, if the Arnold surface $\mathfrak{A}_{\mp}$ is standard, then $\overline{X}^{\pm}$ is decomposable into a connected sum of a few copies of $\mathbb{C}\mathrm{P}^2$, $\overline{\mathbb{C}\mathrm{P}}^2$, if $\mathfrak{A}_{\mp}$ is non-orientable, or copies of $S^2 \times S^2$ if orientable. Such a splitting will be called a complete decomposition. A weaker form of Rokhlin's question: *is it true that $\overline{X}^{\pm}$ admits such a decomposition for any double plane $\mathbb{C}X^{\pm}$ with $\mathbb{R}X^{\pm} \neq \varnothing$.* I will refer to this statement as to *CDQ-Conjecture* (an abbreviation for "Complete Decomposability of the Quotients"). A review of the known result concerning the CDQ-Conjecture and its natural extensions to the other real algebraic surfaces can be found in [F2].

A weaker version of the CDQ-Conjecture was suggested by Akbulut [Ak]: *to prove that SW (Seiberg-Witten) invariants of $\overline{X}^{\pm}$ vanish if $b_2^+(\overline{X}^{\pm}) > 1$.* As an intermediate variation, I present below one more conjecture. Let us call 4-manifolds X and Y *blow-up stable equivalent* (BUS-equivalent) if $X\#_n\overline{\mathbb{C}\mathrm{P}}^2$ is diffeomorphic to $Y\#_m\overline{\mathbb{C}\mathrm{P}}^2$ for some $n, m \geqslant 0$. If X is BUS-equivalent to $\#_n\mathbb{C}\mathrm{P}^2$, $n \geqslant 0$, then we call X *BUS-trivial.* Complete decomposability implies BUS-triviality, but not vice versa (some algebraic surfaces with the inverted orientation are BUS-trivial, as follows from [MM], but not decomposable). On the other hand, BUS-triviality guarantees vanishing of the Donaldson and SW-invariants (more generally, vanishing of these invariants is a property which is preserved under BUS-equivalence, cf. [FS]). So, *the BUS-Conjecture* suggesting that the quotients $\overline{X}^{\pm}$ are BUS-trivial for the double planes, $\mathbb{C}X^{\pm}$ with $\mathbb{R}X^{\pm} \neq \varnothing$, is weaker then CDQ-conjecture, but stronger then Akbulut's conjecture.

1.2. The results. The aim of this paper is to discuss an extension of the Rokhlin Conjecture and its versions to the case of real surfaces with singularities and to present some new results in its support. The main goal is the following theorem.

1.2.1. Theorem. *Assume that a real plane curve, $\mathbb{C}A_0$, of degree $2k$ and with $\mathbb{R}A_0 \neq \varnothing$, splits into a union $\mathbb{C}A_0 = \mathbb{C}B_0 \cup \mathbb{C}C_0$ of transverse non-singular curves and a non-singular curve, $\mathbb{C}A$, is obtained from $\mathbb{C}A_0$ by a small perturbation. Let $\mathbb{C}X^{\pm}$ denote the double plane branched along $\mathbb{C}A$. Then the both quotients, $\overline{X}^{+}$ and $\overline{X}^{-}$, are BUS-trivial. In particular, the SW-invariants of $\overline{X}^{\pm}$ vanish, if $k > 3$.*

The condition $k > 3$ is required for $b_2^+(\overline{X}_t^{\pm}) = \frac{1}{2}(k-1)(k-2) > 1$. The proof of Theorem 1.2.1, which is given in §3, is based on the analysis of the bifurcations which can experience $\overline{X}^{\pm}$ as we deform $\mathbb{C}B_0$ and $\mathbb{C}C_0$. One more condition of vanishing of SW-invariants is given by Theorem 2.3.1.

§2. Rokhlin's Conjecture and its versions in the case of singular real algebraic surfaces

2.1. Smoothly-folding real surface singularities. One or the both of the Arnold surfaces, $\mathfrak{A}_{\pm}$, may turn out to be smoothable even if we admit certain types of real singularities on $\mathbb{C}A$. The condition of smoothability is that the links of $\mathfrak{A}_{\pm} \subset S^4$ at the singular points, $x \in \mathbb{R}A$, are unknots. These unknots are piecewise smooth and their smoothings can be easily extended to smoothings of the embeddings $\mathfrak{A}_{\pm} \subset S^4$. This gives a smoothing of the double covering $\overline{X}^{\pm} \to S^4$, branched along $\mathfrak{A}_{\pm}$, for the double planes, $\mathbb{C}X^{\pm} \to \mathbb{C}\mathrm{P}^2$, branched along $\mathbb{C}A$.

A singularity at a point $x \in \mathbb{R}X$ of a real algebraic surface will be called *smoothly-folding*, or briefly, SF-singularity, if the link of $\overline{X}$ at x is a 3-sphere (so, $\overline{X}$ is a

manifold around x). We call a real algebraic surface $\mathbb{C}X$ *SF-surface* if it has only SF-singularities. Like in the case of the double planes, a smoothing of $\overline{X}$ at the singularities can be easily extended to a smoothing along the whole $\mathbb{R}X$.

A cone-like node (like the locus $\{x^2 + y^2 - z^2 = 0\} \subset \mathbb{C}^3$) is an example of SF-singularity, whereas *a dot-like node* (like the locus $\{x^2 + y^2 + z^2 = 0\}$) is not. Another example of an SF-singularity appears on the double plane $\mathbb{C}X^{\pm}$, if the branching locus, $\mathbb{C}A$, splits at $x \in \mathbb{R}A$ into m non-singular real and pairwise transverse branches, where m is odd. More examples: any simple surface singularity has a real form which is an SF-singularity (I mark this form by a superscript "$-$", e.g., A_n^-, D_n^-, E_n^-).

More generally, SF-singularities can be characterized in terms of a minimal resolution, $\mathrm{res}\colon \mathbb{C}X^{\mathrm{res}} \to \mathbb{C}X$, and the exceptional curve, $\mathbb{C}E \subset \mathbb{C}X^{\mathrm{res}}$, over x (a minimal resolution of $\mathbb{C}X$ yields a real algebraic surface, because it is obtained after several blows up at a real point or at a pair of conj-symmetric imaginary points of a real surface; for more details about real resolutions see, e.g., [S]). Since res may be not *a good* resolution, we have to modify the usual definition of the resolution graph as follows. The vertices of a graph Γ split into 2 types: the vertices of the first type are in 1–1 correspondence with the irreducible components of $\mathbb{C}E$ and the vertices of the other type are in 1–1 correspondence with the intersection points of these components. A pair of vertices of Γ is connected by an edge if one of the vertices represents a component of $\mathbb{C}E$ and the other represents a point on this component. The complex conjugation gives an involution on Γ and we denote by $\overline{\Gamma}$ the quotient graph with respect to this involution.

2.1.1. Theorem. *Assume that $x \in \mathbb{R}X$ is an isolated normal real surface singularity, and $\mathbb{C}E$ is the exceptional curve of its minimal (real) resolution. Then the following conditions are equivalent.*

(1) *x is an SF-singularity,*
(2) *$\overline{E}$ is contractible,*
(3) *$\mathbb{C}E$ splits into real rational irreducible components, $\mathbb{C}E = \mathbb{C}E_1 \cup \dots \mathbb{C}E_n$, with $\overline{E}_i$, $i = 1, \dots, m$, being homeomorphic to a 2-disc D^2, and with the quotient graph $\overline{\Gamma}$ being a tree.*

Remark. If a minimal resolution turns out to be *good*, then we may use the usual resolution graph instead of Γ (in this case, the graph Γ is just a subdivision of the resolution graph).

Note also that the condition $\overline{E}_i \cong D^2$ means that $\mathbb{C}E_i$ is a topological sphere with $\mathbb{R}E_i \neq \varnothing$, or is obtained from such a sphere after identifying a few pairs of conj-symmetric points.

Proof. The equivalence (2)$\Leftrightarrow$(3) is trivial. Let us assume (2) and consider a compact conj-invariant cone-like neighborhood, $\mathbb{C}U \subset \mathbb{C}X$ of x. Let $\mathbb{C}U^{\mathrm{res}} \to \mathbb{C}U$ denote the restriction of the resolution $\mathrm{res}\colon \mathbb{C}X^{\mathrm{res}} \to \mathbb{C}X$ over $\mathbb{C}U$. Note that $\mathbb{C}U^{\mathrm{res}}$ is a regular neighborhood of $\mathbb{C}E$ in $\mathbb{C}X^{\mathrm{res}}$ (i.e., $\mathbb{C}E$ is a spine of $\mathbb{C}U^{\mathrm{res}}$). This implies that $\overline{U}^{\mathrm{res}}$ is a regular neighborhood of $\overline{E}$, and thus, of the graph $\overline{\Gamma}$ (which is a simple deformational retract of $\overline{E}$). Therefore, $\overline{U}^{\mathrm{res}} \cong D^4$ and $\partial(\overline{U}) = \partial(\overline{U}^{\tau}) \cong S^3$.

Assuming (1), we can easily obtain, using the exact homotopy (or homology) sequence of the pair $(\overline{U}^{\mathrm{res}}, \partial(\overline{U}^{\mathrm{res}}))$, that $\overline{U}^{\mathrm{res}}$, and therefore its spine $\overline{E}$, is simply connected; here we do not even use that the resolution, res, is *minimal*. It follows that $\overline{\Gamma}$ is a tree and that all the irreducible components of $\mathbb{C}E$ are rational and

their quotients in $\overline{E}$ can be homeomorphic either to D^2 (if a component is real, i.e., conj-invariant) or to S^2 (if it is not real). To complete the proof, we need to show that all the components of $\mathbb{C}E$ are real.

Blowing up $\mathbb{C}U^{\mathrm{res}}$, we can obtain a real *good resolution*, $\mathbb{C}U^{\mathrm{res}}_g \to \mathbb{C}U$, with the exceptional curve, $\mathbb{C}E_g \subset \mathbb{C}U^{\mathrm{res}}_g$, consisting of non-singular irreducible components, which intersect pairwise transversally and do not have triple intersection points. We can associate graphs Γ_g, $\overline{\Gamma}_g$, to $\mathbb{C}E_g$ like Γ and $\overline{\Gamma}$ were associated to $\mathbb{C}E$.

Let us remove from $\overline{\Gamma}_g$ the vertices corresponding to the real (i.e., conj-invariant) components of $\mathbb{C}E_g$ and to the real intersection points on these components, together with the adjacent edges. The rest splits into several components, $C_1, \dots, C_n$, each component being a tree. The vertices of the first type of C_i represent a set of transversal spheres embedded into $\overline{U}^{\mathrm{res}}_g$. A regular neighborhood of their union is a plumbing manifold, which will be denoted by V_i. Note that the intersection form in V_i is negative, since the form in $\mathbb{C}U^{\mathrm{res}}_g$ (and, thus, in $\overline{U}^{\mathrm{res}}_g$) is negative. On the other hand, it is not difficult to see that $\overline{U}^{\mathrm{res}}_g$ is homeomorphic to a boundary-connected-sum of $V_1, \dots, V_n$, which implies that $\partial V_i \cong S^3$, $i = 1, \dots, n$, since $\partial(\overline{U}^{\mathrm{res}}) \cong S^3$. Now we apply Neumann's result [N] stating that the boundary of a plumbing manifold with a negative intersection form determines this manifold up to blowing up and blowing down. This implies that all the irreducible imaginary components of $\mathbb{C}E_g$ can be blown down and do not appear in $\mathbb{C}E$. $\square$

It is trivial that a blow-up at a non-singular point of $\mathbb{R}X$ does not change $\overline{X}$ (the proof: $\overline{X}' \cong (\mathbb{C}X \# \overline{\mathbb{C}\mathrm{P}}^2)/\operatorname{conj} \cong \overline{X} \# S^4 \cong \overline{X}$). The following observation generalizes this fact.

2.1.2. Corollary. *Assume that $\mathbb{C}X' \to \mathbb{C}X$ is obtained by blowing up at an SF-singularity. Then $\overline{X}' \cong \overline{X}$.*

The proof is not a difficult exercise, after Theorem 2.1.1. $\square$

2.2. The case of curves $\mathbb{C}A$ splitting into real lines. In [F1] I proved the Rokhlin Conjecture in the following special cases.

2.2.1. Theorem. [F1, Theorem 3.1]. *Assume that a curve $\mathbb{C}A_0 \subset \mathbb{C}\mathrm{P}^2$ splits into a generic arrangement of $2k$ real lines, $k \geqslant 1$ and $\mathbb{C}A$ is obtained by a real perturbation of $\mathbb{C}A_0$. Then the both Arnold surfaces, $\mathfrak{A}^\pm$, are standard. In particular, $\overline{X}^\pm$ is completely decomposable for the double planes, $\mathbb{C}X^\pm \to \mathbb{C}\mathrm{P}^2$, branched along $\mathbb{C}A$.*

The Rokhlin Conjecture holds for the Arnold surfaces, $\mathfrak{A}^\pm_0$ of the curve $\mathbb{C}A_0$, as well, and the above theorem indeed implies it, because for a certain canonical perturbation scheme the curve $\mathbb{C}A$ have the Arnold surface $\mathfrak{A}^\pm$ isotopic to $\mathfrak{A}^\pm_0$ (such a canonical perturbation scheme for $\mathfrak{A}^+$ is the opposite to a canonical perturbation scheme for $\mathfrak{A}^-$). A change in the canonical perturbation scheme at any node of $\mathbb{C}A_0$, effects to $\mathfrak{A}^\pm$ as a surgery, which replaces a small 2-disc around the nodal point by a Möbius band, which effects to $\overline{X}^\mp$ as a blow up (here $\mathbb{C}X^\mp$ is, as usual, the corresponding double plane).

The arguments in [F1] are applicable also for real curves on a quadric, $\mathbb{C}A \subset \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$, of bi-degree $(2k, 2l)$, which are obtained by a perturbation from a singular curve, $\mathbb{C}A_0$, splitting into $2k + 2l$ generating lines of the quadric. By blowing up at a real point and the further blowings down, the quadric $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$

is transformed into $\mathbb{C}\mathrm{P}^2$ and the curve $\mathbb{C}A_0$ becomes a configuration of $2k+2l$ real lines, $2k$ of which pass through one point and $2l$ pass through another one.

The method of [F1] is applicable, more generally, for curves $\mathbb{C}A$ splitting into an arbitrary arrangement of $2k$ distinct real lines. Let $\mathbb{C}\mathrm{Q} \to \mathbb{C}\mathrm{P}^2$ be obtained by blowing up at the singular points of $\mathbb{C}A$ whose multiplicity is even and greater then 2, and $\mathbb{C}B \subset \mathbb{C}\mathrm{Q}$ the proper image of $\mathbb{C}A$. Note that $\overline{\mathrm{Q}} \cong S^4$, since a blow up at a real point does not change the quotient. $\mathbb{R}\mathrm{B}$ divides $\mathbb{R}\mathrm{Q}$ into a pair of regions, $\mathbb{R}\mathrm{Q}_\pm$, like in the case of $\mathbb{R}\mathrm{P}^2$, and we can similarly define a pair of the Arnold surfaces $\mathfrak{A}_\pm = \overline{B} \cup \mathbb{R}\mathrm{Q}_\pm \subset S^4$. Furthermore, we can similarly define the double covering, $\mathbb{C}X^\pm \to \mathbb{C}Q$, branched along $\mathbb{C}B$, endowed with one of the two liftings of the complex conjugations on $\mathbb{C}Q$; the induced projection $\overline{X}^\pm \to \overline{Q} = S^4$ is again a double covering branched along $\mathfrak{A}_\mp$. Note that $\mathfrak{A}_\pm$ is connected, unless $\mathbb{C}A$ is a pencil, i.e., unless all the $2k$ lines have a common point. In the case of a pencil, $\mathfrak{A}_\pm$ is a union of 2-spheres which bound disjoint 3-balls in $\overline{\mathrm{Q}}$ (this follows from [F1,Theorem 5.2]), and, thus, $\overline{X}^\pm$ is diffeomorphic to a connected sum $\#_{k-1}(S^1 \times S^3)$.

2.2.2 Theorem. *Assume that $\mathbb{C}A \subset \mathbb{C}\mathrm{P}^2$ splits into a configuration of $2k$ real lines different from a pencil. Then $\mathfrak{A}_+$ and $\mathfrak{A}_-$ are standard surfaces in $S^4 = \overline{\mathrm{Q}}$ and therefore $\overline{X}^+$ and $\overline{X}^-$ are completely decomposable.*

The proof is analogous to the proof of Theorem 3.1 in [F1].

2.3. Vanishing of SW-invariants for $\overline{X}$ after a perturbation of an SF-singularity.

Assume that $\mathbb{C}X$ is a real surface having one singular point, $x \in \mathbb{R}X$, and a non-singular surface $\mathbb{C}X'$ is obtained from $\mathbb{C}X$ by a real perturbation. More precisely, we assume that there exists an equivariant (with respect to the complex conjugation) diffeomorphism between the complement $\mathbb{C}X - \mathrm{Int}(\mathbb{C}U)$ for a compact regular cone-like neighborhood, $\mathbb{C}U \subset \mathbb{C}X$, of x, and the complement $\mathbb{C}X' - \mathrm{Int}(\mathbb{C}U')$ of some compact conj-invariant codimension 0 submanifold $\mathbb{C}U' \subset \mathbb{C}X'$ of a smooth real algebraic surface $\mathbb{C}X'$. A standard example that I mean is $\mathbb{C}U'$ being a Milnor fiber in the case of a complete intersection singularity (for instance, $\mathbb{C}X$ may be a double plane branched along a curve, $\mathbb{C}A$, with one singular point, and $\mathbb{C}X'$ a double plane branched along a non-singular curve $\mathbb{C}A'$ obtained by a perturbation of $\mathbb{C}A$).

2.3.1. Theorem. *Assume that $\mathbb{C}X$ and $\mathbb{C}X'$ are as above, $x \in \mathbb{R}X$ is an SF-singularity and $0 < p_g(\mathbb{C}X^{\mathrm{res}}) < p_g(\mathbb{C}X')$, where p_g denotes the geometric genus and $\mathbb{C}X^{\mathrm{res}} \to \mathbb{C}X$ a real resolution. Then SW-invariants of $\overline{X}'$ vanish.*

Proof. $\partial(\overline{U}) \cong S^3$, for SF-singularities, thus, $\overline{X}'$ is obtained by taking a connected sum of $\overline{X}$ with a 4-manifold $\hat{U}'$, obtained by attaching a 4-ball to $\overline{U}'$.

Standard and well known calculations imply that $b_2^+(\overline{X}') = p_g(\mathbb{C}X')$ and $b_2^+(\overline{X}) = p_g(\mathbb{C}X^{\mathrm{res}})$ (see for instance Lemma 4.2.2 in [F3]), so, $\overline{X}'$ have trivial SW invariants, as a connected sum of manifolds with non-negative intersection forms. $\square$

Remark. The above arguments can be obviously applied as well to SF-surfaces with several singular points, provided the same inequalities hold.

Note also that there exist vanishing theorems for SW-invariants, which generalize the connected sum theorem that was used. Namely, such theorems concern 4-

manifolds which can be split along some other types of 3-manifolds (rather then along S^3, like in the case of the connected sums). Accordingly, Theorem 2.3.1 can be modified and extended to the corresponding class of singularities.

2.4. The singular and the local versions of the Rokhlin problem. A connected sum splitting of $\overline{X}'$ in the above proof of Theorem 2.3.1 gives rise to the questions about the topology of $\mathbb{C}X$ and $\mathbb{C}U'$; for instance, is it true that $\overline{X}$ and $\hat{U}'$ are always completely decomposable whenever they are simply connected ?

Example. Assume that $\mathbb{C}A$ is *almost a pencil*, that is, a union of $2k$ real lines, which all, except one, contain a common point. Then $\mathfrak{A}_\pm$ is an unknotted sphere and $\overline{X}^\pm$ is diffeomorphic to S^4. This shows equivalence of the "local version" of the CDQ-conjecture with the usual "global version", in the particular case of the real double planes, whose branching locus is obtained from $\mathbb{C}A$ by a perturbation.

§3. Proof of Theorem 1.2.1

3.1. Simple nodal deformations of the real plane curves and the double planes.
Recall that the real plane curves of degree d constitute a real projective space, $\mathcal{C}_d$, of dimension $\binom{d+2}{2}$, where the singular curves form a hypersurface, $\Delta \subset \mathcal{C}_d$, called the discriminant hypersurface. By a real deformation of a curve I mean a continuous family of curves, $\mathbb{C}A_t \subset \mathbb{C}\mathrm{P}^2$, $t \in [a, b]$, that is a path in $\mathcal{C}_d$. If a path is generic, it intersects Δ transversally at non-singular points, which are represented by the curves having a nodal singularity (an ordinary double point). In this case, $\mathbb{C}A_t$, $t \in [a, b]$, is called *a simple nodal deformation*.

There may be two types of nodes on real curves: dot-like nodes, like the locus $\{x^2 + y^2 = 0\} \subset \mathbb{C}^2$, and cross-like nodes, like $\{x^2 - y^2 = 0\}$. Assume that d is even and $\mathbb{C}A_t$, $t \in [0, 1]$, is a simple nodal deformation. Taking the double planes branched along $\mathbb{C}A_t$, we obtain two families, $\mathbb{C}X_t^\pm$. Let us fix one of these two continuous families and denote it by $\mathbb{C}X_t$, omitting the superscript in the notation to avoid an ambiguity, which appears if $\mathbb{C}X_0^+$ is continuously transformed into $\mathbb{C}X_1^-$ (this may happen because the sign-superscripts are not defined canonically for nodal curves, like for non-singular ones). Furthermore, I denote by W_t the corresponding region, $\mathbb{R}\mathrm{P}^2_+$ or $\mathbb{R}\mathrm{P}^2_-$, namely, $W_t = p_t(\mathbb{R}X_t)$, where $p_t \colon \mathbb{C}X_t \to \mathbb{C}\mathrm{P}^2$ is the branched covering. The choice of $\mathbb{C}X_t$ is obviously determined by the choice of W_0.

The cross-like nodes of $\mathbb{R}A_t$ are covered obviously by the cone-like nodes of $\mathbb{R}X_t$. A dot-like node at $x \in \mathbb{R}A_t$ is covered by a cone-like node of $\mathbb{R}X_t$ provided x lies in the interior of W_t, otherwise it is covered by a dot-like node of $\mathbb{R}X_t$. If in a process of deformation a cone-like node appears on $\mathbb{R}X_t$, then one of the quotients, $\overline{X}_{t+\varepsilon}$ or $\overline{X}_{t-\varepsilon}$, for $0 < \varepsilon << 1$, is diffeomorphic to $\overline{X}_t$ and the other one differs by a blow-up, i.e., is diffeomorphic to $\overline{X}_t \# \overline{\mathbb{C}\mathrm{P}}^2$. In the case $\mathbb{R}X_t$ has a dot-like node, $\overline{X}_t$ is not a manifold and $\overline{X}_{t+\varepsilon}$ differs from $\overline{X}_{t-\varepsilon}$ by a rational blow-down (or a rational blow-up) of multiplicity 2, in the sense of [FS] (see [Le], [Ak] or [F1]). So, the difficulty to prove that for *an arbitrary* non-singular double plane, $\mathbb{C}X$, the quotient $\overline{X}$ is BUS-trivial consists in proving that its branching locus, $\mathbb{C}A \subset \mathbb{C}\mathrm{P}^2$, can be connected by a *nice* nodal deformation with some other curve, $\mathbb{C}A'$, with a BUS-trivial quotient $\overline{X}'$ of the corresponding double plane $\mathbb{C}X'$. Here, a nodal deformation is *nice* if dot-like nodes do not appear on $\mathbb{R}X_t$.

3.2. Simultaneous nodal deformations. One can handle with the above difficulty if a curve $\mathbb{C}A$ is obtained after a non-singular perturbation of a curve, $\mathbb{C}A_0$, splitting into a union $\mathbb{C}B_0 \cup \mathbb{C}C_0$ of two transverse non-singular real curves. In this case, we consider a deformation $\mathbb{C}A_t$, $t \in [0,1]$, which splits as $\mathbb{C}A_t = \mathbb{C}B_t \cup \mathbb{C}C_t$, where $\mathbb{C}B_t$ and $\mathbb{C}C_t$ are simple nodal deformations. In addition to the nodes, we allow $\mathbb{C}A_t$ to have real singularities of the type A_3, which appear at the points of a simple (quadratic) tangency of $\mathbb{R}B_t$ and $\mathbb{R}C_t$. In a generic pair of nodal deformations, no other singularities are possible (so, none of the curves, $\mathbb{C}B_t$, $\mathbb{C}C_t$, passes through the singularities of the other one, there is no higher order tangency points and no imaginary tangency points). Such a deformation, $\mathbb{C}A_t$, is to be called *a simultaneous nodal deformation.* If moreover, the fixed covering family of the double planes, $\mathbb{C}X_t$, contains no dot-like nodes, then we call $\mathbb{C}A_t$, *a nice simultaneous nodal deformation.*

Given such a deformation, the quotients $\overline{X}_t$, $t \in [0,1]$, appear to be smooth 4-manifolds in the complement of a few singular points. A singular point, $x \in \overline{X}_t$, may appear either from a conjugated pair of imaginary intersection points of $\mathbb{C}B_t$ and $\mathbb{C}C_t$, or from a tangency point of $\mathbb{R}B_t$ and $\mathbb{R}C_t$. In the first case, x is obviously again a nodal singularity. In the second case, a singularity of $\mathbb{C}X_t$ of the complex type A_3 may have two real forms: A_3^+, or A_3^-, the first of which can be represented by the model $\{x^4 - y^2 = z^2\} \subset \mathbb{C}^3$, and the second by the model $\{x^4 - y^2 = -z^2\}$. If $x \in \mathbb{C}X_t$ has type A_3^-, then $\overline{X}_t$ is smoothed at x (since A_n^- is an SF-singularity). If $x \in \mathbb{C}X_t$ has type A_3^+, then the link of $\overline{X}_t$ at x is $\mathbb{R}\mathrm{P}^3$, so $\overline{X}_t$ has at x a nodal singularity.

3.3. Smooth nodal 4-manifolds. By a smooth manifold with isolated singularities I mean a topological space, Y, which is a smooth manifold outside the singular locus, $\Sigma \subset Y$, constituted by a discrete set of points, each of which has a smooth cone-like neighborhood. Such a neighborhood around $y \in \Sigma$ is by definition a compact neighborhood, $U_y \subset Y$, endowed with a homeomorphism, $\phi_y \colon U_y \to \mathrm{Con}(L_y)$, which is a diffeomorphism outside y. Here L_y is a 3-manifold called *the link of Y at y* and $\mathrm{Con}(L_y)$ is a cone with the base L_y (i.e., a join of L_y and y). We may view (U_y, ϕ_y) as a "chart" required to bring a "smooth" structure to y. It is well known that algebraic varieties with isolated singularities have such a structure. The quotients $\overline{X}$ of real algebraic surfaces, $\mathbb{C}X$, having only isolated singularities, give another example. We consider below only a special case of smooth 4-manifolds with all the links L_y being homeomorphic to $\mathbb{R}\mathrm{P}^3$; such manifolds will be called *smooth nodal 4-manifold.*

Given such a manifold, Y, we define a *resolution* of a singularity at $y \in Y$, as a surgery replacing a cone-like neighborhood, U_y, by the total space of a smooth D^2-fiber bundle over S^2 with the normal Euler number -2 (that is the usual surgery characterizing a resolution of an algebraic node). Let us denote by Y^{res} the manifold obtained after resolution of all the nodes of Y. The differential type of Y^{res} is well defined, because the orientation-preserving diffeomorphism group of $\mathbb{R}\mathrm{P}^3$ is known to be connected (cf. [H]). We call a pair of smooth nodal 4-manifolds, Y_1, Y_2, BUS-equivalent, if Y_1^{res} is BUS-equivalent to Y_2^{res}. We call such a nodal manifold, Y, BUS-trivial, if Y^{res} is BUS-trivial.

3.4. Proof of Theorem 1.2.1. Theorem 1.2.1 follows from the lemmas below.

3.4.1. Lemma. *Assume that $\mathbb{C}A_t = \mathbb{C}B_t \cup \mathbb{C}C_t$, $t \in [0,1]$, is a nice simultaneous*

nodal deformation of plane real curves, of degree $\deg(\mathbb{C}A_t) = 2k$, $k \geqslant 1$, *and* $\mathbb{C}X_t$ *the associated deformation of the double planes. Then* $\overline{X}_0$ *is BUS-equivalent to* $\overline{X}_1$.

Here, as before, $\mathbb{C}X_t$ is the corresponding family of the double planes, which is determined by fixing a choice of W_0.

3.4.2. Lemma. *Assume that plane real curves* $\mathbb{C}A_0$ *and* $\mathbb{C}A_1$ *of degree* $2k$, $k \geqslant 1$, *with* $\mathbb{R}A_0 \neq \varnothing$, $\mathbb{R}A_1 \neq \varnothing$, *split into unions* $\mathbb{C}A_0 = \mathbb{C}B_0 \cup \mathbb{C}C_0$ *and* $\mathbb{C}A_1 = \mathbb{C}B_1 \cup \mathbb{C}C_1$ *of nonsingular transverse curves, so that* $\mathbb{C}B_0$ *and* $\mathbb{C}B_1$ *have the same degree. Then* $\mathbb{C}A_0$ *and* $\mathbb{C}A_1$ *can be connected by a nice (with respect to any fixed choice of* W_0*) simultaneous nodal deformation.*

In combination with Theorem 2.2.1 this gives

3.4.3. Corollary. *Assume that a curve* $\mathbb{C}A_0$ *splits like in Lemma 3.4.2. above. Then* $\overline{X}_0$ *is BUS-trivial.*

Perturbing the imaginary singularities of $\mathbb{C}A_0$ we resolve the nodal singularities of $\overline{X}_0$; perturbing the real nodes we blow up $\overline{X}_0$ or preserve it topologically, as was mentioned in 3.1. So, after perturbation of all the nodes of $\mathbb{C}A_0$ we obtain a BUS-equivalent (thus, a BUS-trivial) manifold. This completes the proof of Theorem 1.2.1. □

Remark. The condition that the curves $\mathbb{C}B_0$ and $\mathbb{C}C_0$ are transverse (like a generic position condition that the tangency points of $\mathbb{C}B_t$ and $\mathbb{C}C_t$ should not have order higher then 2) is not very essential in Theorem 1.2.1. This follows from analysis of deformations of a singularity at a point $s \in \mathbb{C}A_0$, at which $\mathbb{C}B_0$ and $\mathbb{C}C_0$ have an oder n tangency (that is a simple singularity of the type A_{2n-1}). More generally, using the well-known classification of deformations of the real simple singularities (cf. [Ch]), one can prove the following

3.4.4. Theorem. *If a real surface* $\mathbb{C}X$ *has a simple singularity at* $s \in \mathbb{R}X$ *which is not equivalent to the singularity defined in* $\mathbb{C}^3$ *by the equation* $x^{2n} + y^2 + z^2 = 0$, *and* $\mathbb{C}X'$ *is obtained by a real non-singular perturbation of* $\mathbb{C}X$, *then the BUS-equivalence class of* $\overline{X}'$ *is independent of the perturbation.*

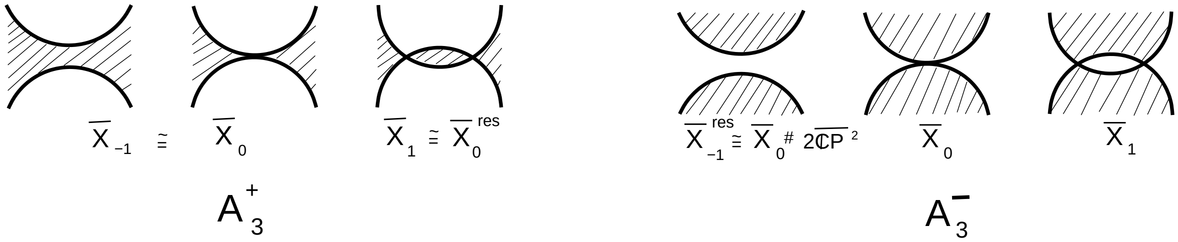
For the above exceptional singularity, the same claim is true for all the perturbations except the one for which the real locus, $\mathbb{R}X'$, *vanishes near* s.

In fact, $\overline{X}$ has at s a singularity topologically equivalent to an algebraic surface singularity and $\overline{X}'$ is BUS-equivalent to $\overline{X}^{\mathrm{res}}$, obtained from $\overline{X}$ by resolution of this singularity. There exists a conceptual proof of it, which explains this phenomenon better then a straightforward routine analysis. It uses a version of the trick applied by Donaldson [D] to real K3 surfaces. This trick consists in variation of the complex structure (within a family parameterized by a twistor line), so that the involution of complex conjugation becomes holomorphic. One can apply this trick similarly for real simple singularities and their real deformations, using the Kronheimer Torelli-like theorem [Kr] instead of Yau's theorem used by Donaldson.

3.5. Proof of Lemma 3.4.1. It is enough to determine the bifurcation of $\overline{X}_t$, after passing a singularity of the type $A_3^{\pm}$ (see Figure 1).

Since $\overline{X}_t \to S^4$ is a double covering branched along the corresponding Arnold surface, $\mathfrak{A}_t$, for $\mathbb{C}A_t$, the problem is reduced to analysis of the bifurcations of $\mathfrak{A}_t$ in the following model example.

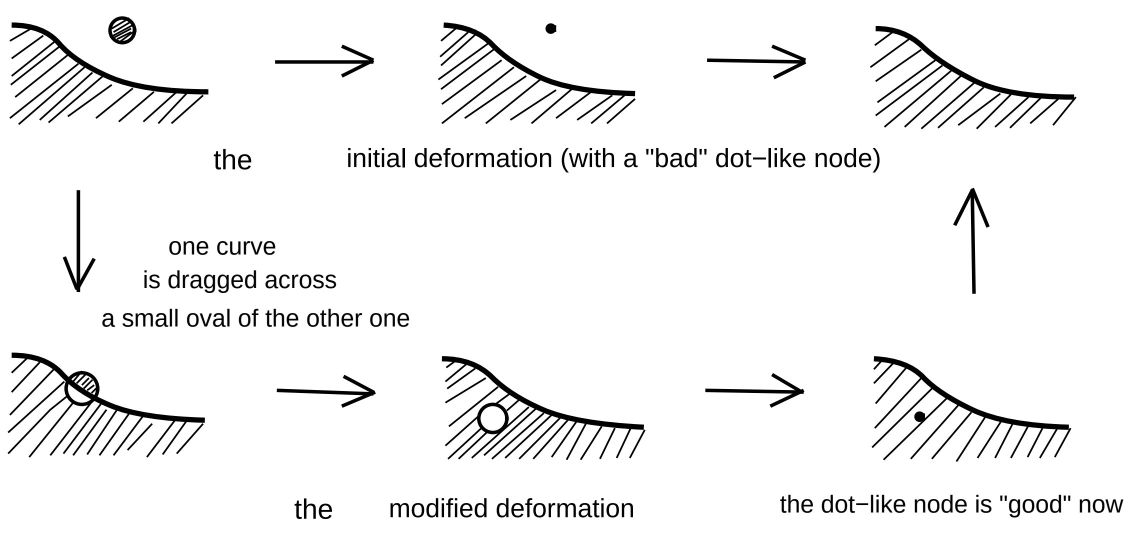
FIGURE 1. The singularities A_3^+ and A_3^-. The region W_t is shaded.



Let $\mathbb{C}A_t \subset \mathbb{C}^2$, $t \in [-1, 1]$, denote the zero locus $\{f_t(x, y) = 0\}$, where $f_t(x, y) = (x^2 - t)^2 - y^2$. The family of surfaces $\mathbb{C}X_t \subset \mathbb{C}^3$ defined by the equation $f_t(x, y) = z^2$ represents a deformation of a singularity A_3^+. In this case, the quotient $\overline{X}_t \cong \overline{X}_0$ has one node for $t \leqslant 0$, and $\overline{X}_t \cong \overline{X}_0^{\mathrm{res}}$, for $t > 0$. A deformation of a singularity A_3^- is represented by a family of surfaces $\mathbb{C}X_t = \{f_t(x, y) = -z^2\} \subset \mathbb{C}^3$. In this case, $\overline{X}_t \cong \overline{X}_0$ is smooth, if $t \geqslant 0$. If $t < 0$, then $\overline{X}_t$ has a node, such that $\overline{X}_t^{\mathrm{res}} \cong \overline{X}_0 \# 2\overline{\mathbb{C}\mathrm{P}}^2$. Thus, in the both cases, $\overline{X}_{-1}$ is BUS-equivalent to $\overline{X}_1$. $\square$

3.6. Proof of Lemma 3.4.2. We can make "nice" a given simultaneous nodal deformation, $\mathbb{C}A_t = \mathbb{C}B_t \cup \mathbb{C}C_t$ using the following trick. Assume that $\mathbb{C}A_{t_0}$ has a "bad" dot-like node, $x \notin \mathrm{Int}(W_{t_0})$, and for definiteness, $x \in \mathbb{R}B_{t_0}$. Let us assume $\mathbb{C}C_{t_0}$ is non-singular (just a generic position assumption) and that $\mathbb{R}C_{t_0} \neq \varnothing$ (a more essential assumption). Choose a sufficiently small $\varepsilon > 0$, so that neither $\mathbb{C}B_t$, nor $\mathbb{C}C_t$ have singularities for $|t - t_0| \leqslant \varepsilon$, except the one at x for $t = t_0$, and furthermore, so that $x \notin \mathbb{R}C_t$ for $|t - t_0| \leqslant \varepsilon$. We may choose a loop, $[t_0 - \varepsilon, t_0 + \varepsilon] \to PGL(3; \mathbb{R})$, $t \mapsto T_t$, centered at the unity, and replace $\mathbb{C}C_t$, $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, by a curve $T_t(\mathbb{C}C_t)$ obtained from $\mathbb{C}C_t$ by the action of T_t (i.e., just move $\mathbb{C}C_t$ by linear projective transformations), so that the real part $\mathbb{R}C_t$ passes once across the point $x \in \mathbb{R}\mathrm{P}^2$ as t vary from $t_0 - \varepsilon$ to t_0, see Figure 2.

FIGURE 2. Elimination of a "bad" dot-like nodal singularity (W_t is shaded)



For a generic choice of the family T_t, this modification gives again a simultaneous nodal deformation, but now a dot-like node at x appears in the interior of the

(modified) region W_{t_0} (see Figure 2).

To apply the above trick and get rid of all the "bad" nodes, it is required that $\mathbb{C}C_t \neq \varnothing$ if a "bad" node appears on $\mathbb{C}B_t$ and $\mathbb{C}B_t \neq \varnothing$ if such a node appears on $\mathbb{C}C_t$. It is not difficult to provide this as follows. Assuming that $\mathbb{R}B_0 \neq \varnothing$ and $\mathbb{R}C_1 \neq \varnothing$, we deform first $\mathbb{C}C_0$ to obtain $\mathbb{C}C_1$, not varying $\mathbb{C}B_0$ (allowing though linear projective transformations, if needed to guarantee a generic position with $\mathbb{C}C_1$), and then deform $\mathbb{C}B_0$ to obtain $\mathbb{C}B_1$. The case $\mathbb{R}C_0 \neq \varnothing$ and $\mathbb{R}B_1 \neq \varnothing$ is analogous. If $\mathbb{R}C_0 = \mathbb{R}C_1 = \varnothing$, but $\mathbb{R}B_i \neq \varnothing$, $i = 0, 1$, then we first connect $\mathbb{C}C_0$ by a simple nodal deformation with any auxiliary non-singular curve $\mathbb{C}C_0'$ with $\mathbb{R}C_0' \neq \varnothing$, then deform $\mathbb{C}B_0$ to obtain $\mathbb{C}B_1$, and finally, connect $\mathbb{C}C_0'$ by a deformation with $\mathbb{C}C_1$. The case $\mathbb{R}B_0 = \mathbb{R}B_1 = \varnothing$ is analogous. □

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